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# Bifurcation of traveling waves resulting from resonant mode crossings in oceanic currents

**M. RENARDY**

ABSTRACT. – We consider parallel currents in a rotating system. The current has a finite width and is bounded by density fronts on both sides. It has been shown that such flows have instabilities associated with a 1:1 resonance, i.e. the merging of two real wave speeds which then become complex. In this paper, we analyze bifurcations resulting from these instabilities. Two types of bifurcations arise; one is a reversible Hopf bifurcation, and the other is a reversible Hopf bifurcation with $O(2)$-symmetry. © Elsevier, Paris.

## 1. Introduction

In [2], Griffiths et al. investigate the stability of gravity currents in a rotating system. The flow geometry is infinitely long in the direction of the flow and the current depth vanishes on both sides of the flow. A model is used where only a layer of heavy fluid at the bottom is flowing, while the lighter fluid on the outside is at rest. The balance between hydrostatic pressure and Coriolis forces allows for sharp density fronts on both sides of the bottom current. It is found that these density fronts are unstable, resulting in a meandering motion of the current. Griffiths et al. report on laboratory experiments demonstrating the instability. They cite the flow of cold water through the Denmark strait (which lies between Greenland and Iceland) and along the sloping bottom south of the strait as a potential oceanographic application of the model. The objective of this paper is to analyze the bifurcations associated with the instability.

We begin by describing the model problem. As is customary in oceanographic applications, we use the shallow water approximation and the hydrostatic approximation, i.e. the pressure is determined purely by hydrostatic effects. Within this approximation, the equations result from a balance of inertial forces, hydrostatic pressure and the horizontal component of the Coriolis force. We consider a fluid of uniform density $\rho_2$ flowing beneath an infinite (stagnant) region of fluid of a lighter density $\rho_1$. The flow is on a uniformly sloping bottom with gradient $-\beta$ in the $y$-direction. We denote by $h$ the depth of the bottom layer, and by $u$ and $v$ the velocity components in the $x$ and $y$ directions. Moreover, $g' = g(\rho_2 - \rho_1)/\rho_1$ denotes the reduced gravity, and $f$ denotes the Coriolis parameter. With the assumptions of the shallow water and hydrostatic approximation, the equations of motion are

$$\begin{aligned} u_t + uu_x + vu_y - fv &= -g'h_x, \\ v_t + uv_x + vv_y + fu &= -g'h_y + g'\beta, \\ h_t + (uh)_x + (vh)_y &= 0. \end{aligned} \tag{1}$$

Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, USA.

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We nondimensionalize by scaling $h$ with a typical depth scale $H$, $x$ and $y$ with $(g'H)^{1/2}f^{-1}$, $t$ with $f^{-1}$ and $u$ and $v$ with $(g'H)^{1/2}$. This yields the dimensionless equations

$$\begin{aligned} &u_t + uu_x + vu_y - v = -h_x, \\ &v_t + uv_x + vv_y + (u - U) = -h_y, \\ &h_t + (uh)_x + (vh)_y = 0, \end{aligned} \tag{2}$$

where $U = (g')^{1/2}H^{-1/2}f^{-1}\beta$. Equations (2) have the solution $h = 1 - \frac{1}{2}y^2$, $u = U + y$, $v = 0$, and we are interested in small perturbations of this solution. We note that the flow region is given by $h > 0$, i.e. $|y| < \sqrt{2}$. Since $U$ can be transformed away by a Galilean transformation, we shall henceforth assume it to be zero.

A cross-section of the flow region and the basic depth profile $h$ is shown in Figure 1; the region of flowing liquid is underneath the parabola. The plot is not drawn to scale; in reality the slope $\beta$ is very small, and also the depth scale $H$ is small relative to the horizontal length scale $(g'H)^{1/2}f^{-1}$. The flow is perpendicular to the plane of the plot, and the velocity is a superposition of a uniform speed $U$ going into the paper, and a perturbation which is linear in $y$, going into the paper on the side marked by $x$ and coming out of the paper on the side marked by $o$. The vertical component of the Earth's angular velocity is in the positive $z$-direction. The instabilities discussed in the following are associated with the motion of the fronts at $y = \pm\sqrt{2}$, where the depth of the flowing region approaches zero. The location of these fronts is not fixed; under a perturbation to the flow they will move according to the kinematic condition. The flow instabilities therefore lead to meandering of the bottom current; the boundaries are no longer straight lines and may, for instance, have a snake-like appearance.

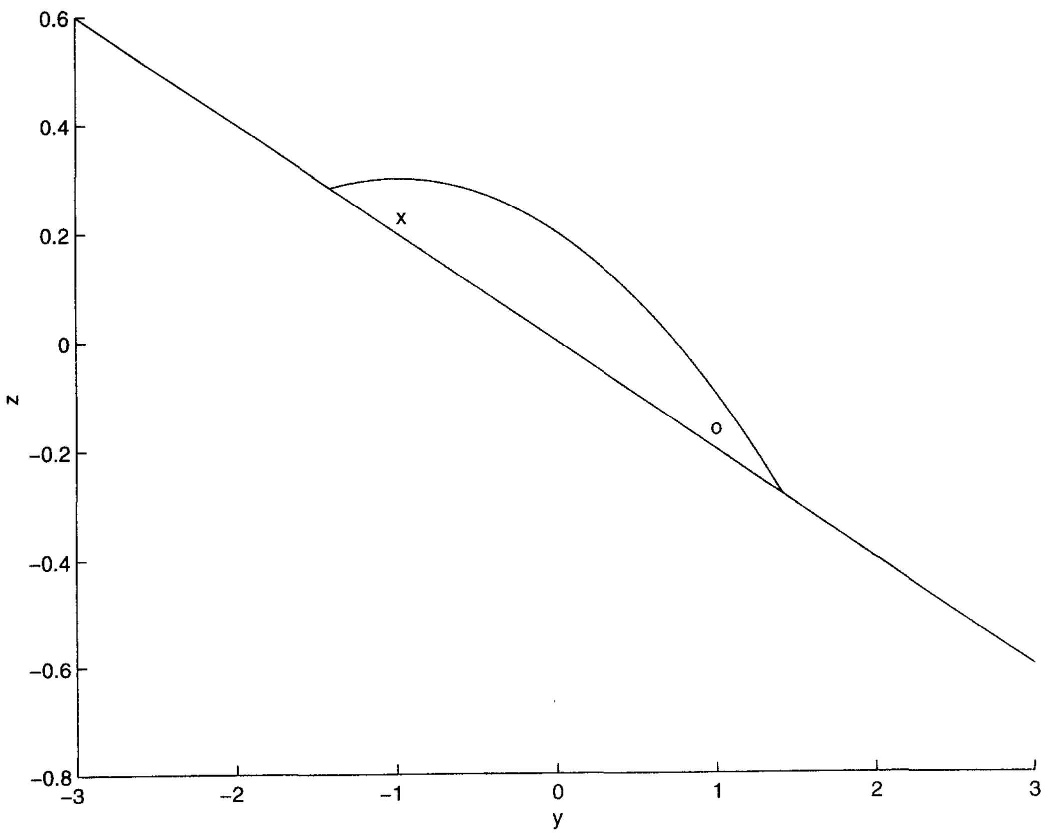


Fig. 1. – Sketch of the flow geometry.

We simplify the equations further by limiting our attention to solutions with zero potential vorticity. For such solutions, we can set

$$u = y + \phi_x, \quad v = \phi_y, \quad h = 1 - \frac{y^2}{2} + \tilde{h}, \tag{3}$$

and the three equations in (2) can be replaced by the two equations

$$\begin{aligned} &\phi_t + y\phi_x + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + \tilde{h} = k(t), \\ &\tilde{h}_t + ((y + \phi_x)\tilde{h})_x + (\phi_y \tilde{h})_y + \left(1 - \frac{y^2}{2}\right)(\phi_{xx} + \phi_{yy}) - y\phi_y = 0, \end{aligned} \tag{4}$$

where $k(t)$ is a function of time which we are free to choose, since it only affects the values of the potential $\phi$ and not the velocities, i.e. the spatial derivatives of $\phi$.

To consider the linear stability, we neglect the quadratic terms in (4) and look for a solution proportional to $\exp(i\alpha x + \sigma t)$. This yields the system

$$\begin{aligned} &(\sigma + i\alpha y)\tilde{\phi} + \tilde{h} = 0, \\ &(\sigma + i\alpha y)\tilde{h} - \left(1 - \frac{y^2}{2}\right)\alpha^2\phi + \left(\left(1 - \frac{y^2}{2}\right)\phi_y\right)_y = 0. \end{aligned} \tag{5}$$

Solutions are to be found for $-\sqrt{2} < y < \sqrt{2}$. Due to the degeneracy of the equation when $1 - y^2/2 = 0$, no boundary conditions are needed. This eigenvalue problem is studied in [2], [3] and [8]. As $\alpha$ varies, there are an infinite number of points where two imaginary eigenvalues merge and then leave the imaginary axis. The goal of this paper is to study the bifurcations associated with this transition. For the bifurcation analysis, we shall assume that the flow is periodic in the $x$-direction, with an a priori given period. The length of this period is our bifurcation parameter.

We note that (4) is time reversible: If $\phi(x, y, t)$, $\tilde{h}(x, y, t)$ is a solution, then $-\phi(-x, y, -t)$, $\tilde{h}(-x, y, -t)$ is also a solution. As a consequence, if $\sigma$ is an eigenvalue of (5) for wavenumber $\alpha$, then $-\sigma$ is an eigenvalue for $-\alpha$. In addition to time reversibility, there is an $O(2)$ symmetry. We have invariance under translations in $x$ and under reflection across the origin $(x, y) \to (-x, -y)$. It follows from this that the eigenvalues of (5) for wavenumbers $\alpha$ and $-\alpha$ must be the same, and consequently the eigenvalues for every fixed wavenumber form complex conjugate pairs. Combining the time reversibility and reflection symmetry, we find that if $\phi(x, y, t)$, $\tilde{h}(x, y, t)$ is a solution, then so is $-\phi(x, -y, -t)$, $\tilde{h}(x, -y, -t)$. A consequence of this is that if $\sigma$ is an eigenvalue of (5) for a wavenumber $\alpha$, then so is $-\sigma$. Hence eigenvalues of (5) can appear only in the following combinations: a zero eigenvalue, a pair $\pm\sigma$ of purely real eigenvalues, a pair of purely imaginary eigenvalues, or a complex quadruplet consisting of $\pm\sigma$ and $\pm\bar{\sigma}$. Transitions to instability occur when either a pair of imaginary eigenvalues merges at the origin and becomes a pair of real eigenvalues or two pairs of imaginary eigenvalues merge and become a complex quadruplet. Both cases occur and will be analyzed.

The first of the two cases is formally equivalent to the reversible Hopf bifurcation as analyzed in [4]. At the bifurcation point, we have a double, nonsemisimple eigenvalue 0 for wave number $\alpha$ and $-\alpha$. If we go back to the laboratory frame and take account of the mean velocity $U$, then we have instead a double, nonsemisimple eigenvalue $-i\alpha U$ at wave number $\alpha$, and a corresponding eigenvalue $i\alpha U$ at wave number $-\alpha$.

In the other case, we have two double, nonsemisimple eigenvalues $\pm i\omega_0$ at wave number $\alpha$ and $-\alpha$. This is a reversible Hopf bifurcation with $O(2)$ symmetry; the bifurcation equation is a problem in eight dimensions.

As usual in bifurcations with symmetry, we can obtain a reduction of the dimension by looking for solutions with certain symmetries; this leads to traveling and standing wave solutions (see [10]).

Figures 2 and 3 show the linear eigenvalues in the range $0 \leq \alpha \leq 5$. Except for a difference of a normalization factor $2\sqrt{2}$ on the horizontal axis, these figures agree with Figure 2 in [3]. The first figure shows the real part of $\sigma$, i.e. the growth rate, for those eigenvalues for which it is nonzero. The second figure shows the wave speed, $\text{Im}\,\sigma/\alpha$. We can see that nonzero growth rates results from crossovers. Whenever two purely imaginary eigenvalues cross, there is a small interval on which they become complex. The pattern continues indefinitely as $\alpha$ is increased (for a proof see [8]), but the growth rates become very small and the intervals on which the eigenvalues are complex become extremely narrow. In the following sections, we shall analyze the bifurcation near $\alpha = 1.04$ where two real eigenvalues merge at zero and become purely imaginary, and the first resonant crossover near $\alpha = 1.916$. The relevant points are marked by crosses on Figure 2, and by circles on Figure 3.

We note that while our problem can be formally treated like the reversible Hopf bifurcation in [4] , there are significant differences. In the situation which Iooss and Pérouème have in mind, the four-dimensional system which they study is a reduced system on a center manifold; that is, the rest of the linearized spectrum is off the imaginary axis. In our problem, in contrast, there are always infinitely many eigenvalues on the imaginary axis. Thus, while it is possible to use a formal multiscale expansion and arrive at a system of amplitude equations like those of [4], a rigorous justification would be very difficult. Even beyond this difficulty, there are challenges associated with the character of the partial differential equations: The physical domain, i.e. the region where $h > 0$, varies with time, and, moreover, the hyperbolic system (4) becomes degenerate for $h = 0$ and actually changes type when $h$ becomes negative. Some rigorous results for the initial-value problem were obtained in [9].

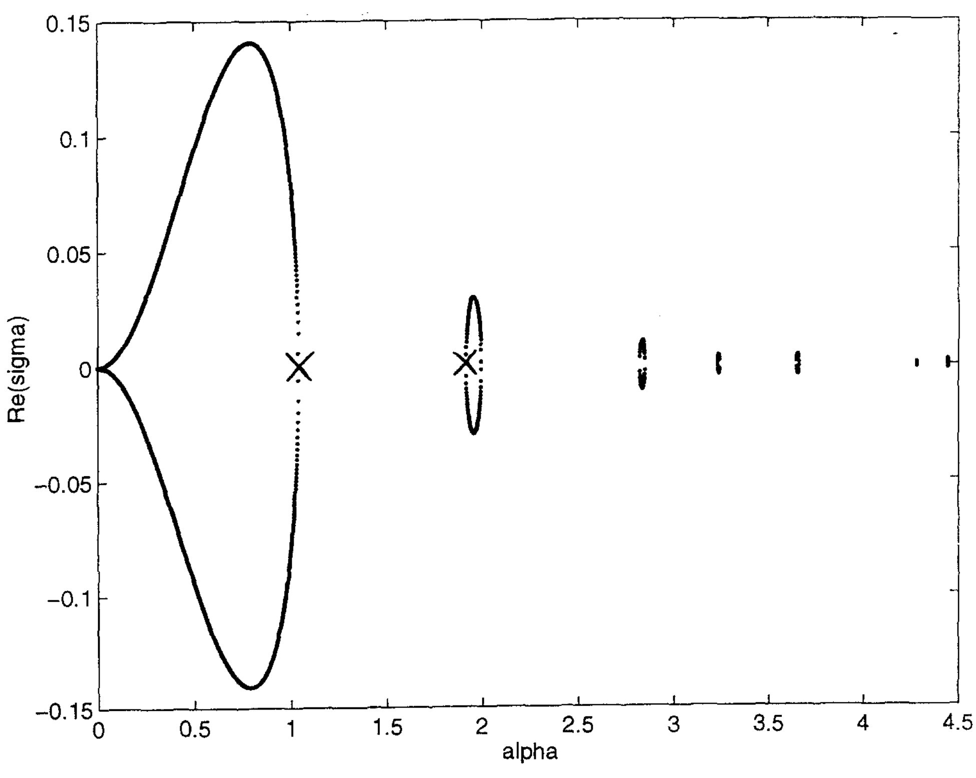


Fig. 2. – Growth rate as a function of wavenumber.

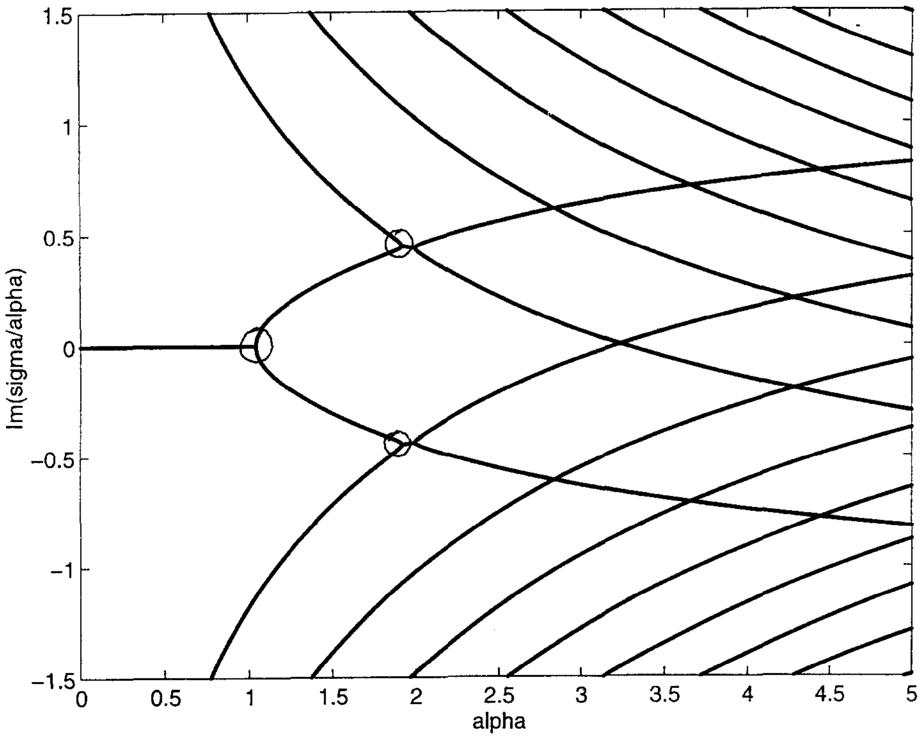


Fig. 3. – Wave speed as a function of wavenumber.

## 2. Reversible Hopf bifurcation

In our present analysis, we derive formal amplitude equations for solutions which are governed by the interaction of a small number of modes, and these modes are given by analytic functions which can be continued smoothly into the region where $h < 0$. This allows us to continue to treat the equations on the interval $-\sqrt{2} < y < \sqrt{2}$.

We look for solutions of (4) which are periodic in $x$ with a given period $L$. We regard $L$ as a bifurcation parameter. As we have seen in the preceding section, there are instabilities for arbitrarily large wavenumbers and hence there can be unstable modes for arbitrarily small values of $L$. However, the most significant growth rates occur for $\alpha < \alpha_0$, where $\alpha_0$ is the value close to 1.04 where two imaginary eigenvalues merge at zero become a real pair. It is reasonable to expect that shorter wavelength instabilities with very small growth rates are not very significant in practice. We shall therefore consider the onset of instabilities as $L$ crosses through the value $L_0 = 2\pi/\alpha_0$.

Throughout our analysis, we shall fix the function $k(t)$ in (4) in such a way that

(6) $$\int_0^L \int_{-\sqrt{2}}^{\sqrt{2}} \left[\frac{1}{2}(\phi_x^2 + \phi_y^2) + \tilde{h}\right] dy\, dx = 2\sqrt{2}Lk(t),$$

and hence

(7) $$\int_0^L \int_{-\sqrt{2}}^{\sqrt{2}} \phi\, dy\, dx$$

is a constant independent of time, which we can arbitrarily set to zero.

We write (4) in the schematic form

$$\dot{u} = A(L)\,u + N(u, u, L), \tag{8}$$

where $u$ represents the pair $(\phi, \tilde{h})$, the operator $A(L)$ represents the linear terms, and $N(u, u, L)$ represents quadratic terms. For the following, we find it convenient to extend the definition of $N$ to unequal arguments in the obvious fashion: $N(u, v, L) = \frac{1}{4}(N(u+v, u+v, L) - N(u-v, u-v, L))$. In order to have a function space independent of $L$, we rescale $x$ and set $\hat{x} = 2\pi x/L$, so we are always looking for $2\pi$-periodic solutions in the rescaled variable. For $L = L_0$, the operator $A(L_0)$ has the eigenvalue 0 with an eigenfunction of the form $\zeta = u_1(y)\exp(i\hat{x})$. Moreover, the eigenvalue is algebraically twofold, and hence there is a function $\xi = u_2(y)\exp(i\hat{x})$ for which $A(L_0)\,\xi = \zeta$. We define adjoint eigenfunctions $c = v_1(y)\exp(i\hat{x})$ and $d = v_2(y)\exp(i\hat{x})$ by the relations

$$A^*(L_0)\,c = 0, \quad A^*(L_0)\,d = 0, \quad (c, \xi) = 1, \tag{9}$$

where $(\cdot,\cdot)$ is an appropriate inner product, and $A^*$ denotes the adjoint of $A$. There is also an eigenfunction with eigenvalue zero and no $x$-dependence, give by $\phi = 0$, $\tilde{h} = 1$. We denote this eigenfunction by $\eta$, and the corresponding adjoint eigenfunction by $f$. We normalize so that $(f, \eta) = 1$. The eigenfunction with zero eigenvalue is associated with a conservation law, namely the conservation of the volume of the heavy fluid, as we shall show in detail below. Hence it is clear that by integrating this conservation law, we can eliminate the corresponding amplitude (denoted by $g$ below) from the set of amplitude equations, i.e. the presence of the zero eigenvalue has no effect on the form of the equations we derive. We choose to retain $g$, because we wish to derive the coefficients of the amplitude equations in a form where they can easily be programmed for numerical evaluation.

We introduce a small parameter $\varepsilon$ and assume that $L - L_0$ is of order $\varepsilon^2$. We seek solutions which are of the form

$$\begin{aligned} u = {} & \varepsilon\,(a(\varepsilon t)\,\zeta + \overline{a}(\varepsilon t)\,\overline{\zeta}) + \varepsilon^2\,(b(\varepsilon t)\,\xi + \overline{b}(\varepsilon t)\,\overline{\xi} \\ & + g(\varepsilon t)\,\eta + a^2\psi + \overline{a}^2\overline{\psi} + a\overline{a}\chi) + O(\varepsilon^3), \end{aligned} \tag{10}$$

where $\psi$ and $\chi$ are terms generated by quadratic interactions. Using appropriate projections of the equation of motion (8), we derive the reduced equations

$$\begin{gathered} \dot{a} = b, \quad \dot{b} = \gamma a + 2|a|^2 a\,(c, N(\psi, \overline{\zeta})) + 2|a|^2 a\,(c, N(\chi, \zeta)) + 2ag\,(c, N(\eta, \zeta)), \\ A(L_0)\,\psi + N(\zeta, \zeta) = 0, \\ A(L_0)\,\chi + 2N(\zeta, \overline{\zeta}) = 0, \quad (f, \chi) = 0, \\ \dot{g} = 2\,(a\overline{b} + b\overline{a})\,(f, N(\zeta, \overline{\zeta})). \end{gathered} \tag{11}$$

Here we have used the fact that $(f, N(\zeta, \overline{\zeta}))$ turns out to be zero, and $(f, N(\zeta, \overline{\xi}))$ is real. We shall discuss the specific form of these terms below. The equations (11) are invariant under translations $(a, b, g) \to (a\exp(i\phi), b\exp(i\phi), g)$, reflections $(a, b, g) \to (\overline{a}, \overline{b}, g)$ and time reversibility $(a, b, g) \to (\overline{a}, -\overline{b}, g)$, $t \to -t$. The coefficient $\gamma$ is defined as

$$\gamma = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\,(c, (A(L) - A(L_0))\,\zeta). \tag{12}$$

We can write the equations (11) in the form

$$(13)\qquad \begin{aligned} \ddot{a} &= \gamma a + \kappa |a|^2 + \nu a g, \\ \dot{g} &= \mu\,(a\dot{\overline{a}} + \dot{a}\overline{a}), \end{aligned}$$

where

$$(14)\qquad \kappa = 2\,(c, N\,(\chi,\zeta)) + 2\,(c, N\,(\psi,\overline{\zeta})), \quad \nu = 2\,(c, N\,(\eta,\zeta)), \quad \mu = 2\,(f, N\,(\zeta,\overline{\xi})).$$

It follows from the symmetries that the coefficients $\gamma$, $\kappa$ and $\nu$ are real.

We can integrate the second equation in (13) to obtain $g = \mu|a|^2 + C$, and we shall set the integration constant $C$ equal to zero; as shown below, this simply means that the volume of the heavy fluid is the same as in the unperturbed base flow. The first equation then becomes

$$(15)\qquad \ddot{a} = \gamma a + (\kappa + \mu\nu)|a|^2 a.$$

At the third order level, this is precisely the equation for the reversible Hopf bifurcation [4]; the same equation arises for the Hamiltonian Hopf bifurcation [5]-[7]. Equation (15) is well studied in the literature; it is a Hamiltonian system with the Hamitonian $|\dot{a}|^2/2 - \gamma|a|^2/2 - (\kappa + \mu\nu)|a|^4/4$, and with the additional integral $i\,(\overline{a}\dot{a} - a\dot{\overline{a}})$ and hence it is completely integrable. Iooss and Pérouème [4] also discuss the normal form for higher order corrections to (15). In the present case, there is the additional reflection symmetry $(a, b) \to (\overline{a}, \overline{b})$ which would remove the $\phi_0$-term in their equation (3), leading to the general form

$$\ddot{a} = a\phi\,(L, |a|^2, (a\dot{\overline{a}} - \dot{a}\overline{a})^2).$$

(This more general equation can also be integrated exactly). The analysis in [4] is focussed primarily on homoclinic solutions; the interest in this paper is in the question whether stable waves exist when $\gamma > 0$, i.e. when the parallel flow solution is unstable. As is well known, this depends on the sign of $\kappa + \mu\nu$.

For traveling wave solutions, we set $a = a_0 \exp(i\omega t)$. We find the equation

$$(16)\qquad (\kappa + \nu\mu)|a_0|^2 = -\omega^2 - \gamma.$$

If $\kappa + \nu\mu < 0$, then solutions exist for all $\omega$ if $\gamma > 0$, i.e. when the linearized system is unstable (supercritical bifurcation). If, on the other hand $\gamma < 0$, then we must have $\omega^2 > -\gamma$. The stationary solution with $\omega = 0$ exists for $\gamma > 0$, and it is the solution of minimum amplitude. If $\kappa + \nu\mu > 0$, we have subcritical bifurcation, i.e. solutions of (14) exist ony for $\gamma < 0$, the stationary solution is the one of maximum amplitude, and the amplitude reaches zero if $\omega^2 = -\gamma$.

Within the context of the amplitude equations (13), the stability of the solutions is determined by the sign of

$$(17)\qquad 3\,(\kappa + \mu\nu)|a_0|^2 + \gamma = -2\gamma - 3\omega^2.$$

This implies that supercritical solutions are stable, while subcritical solutions are stable only if the frequency is sufficiently large (i.e. the amplitude is sufficiently small). In particular, the steady solution with $\omega = 0$ is unstable if it is subcritical.

A Chebyshev collocation method was used to compute $\kappa + \mu\nu$ numerically. The value turns out to be positive, i.e. the bifurcation is subcritical, and no traveling wave solutions should be expected in the unstable regime.

This is in qualitative agreement with the experiments reported in [2], where no saturation in the amplitude of the instability was observed, and the current breaks up into a string of eddies. We can decompose $\kappa + \mu\nu$ into a contribution that results from the interaction between the primary wave and the second harmonic, and a contribution that results from the interaction between the primary wave and the perturbation to the mean flow. Interestingly, the two contributions almost cancel each other. The contribution from the second harmonic is negative (stabilizing), while the contribution from the mean flow is positive (destabilizing).

We refer to the prior literature [2], [3] for visualization of what the perturbed flow looks like; both [2] and [3] give contour plots of eigenfunctions, and [2] also shows photographs of experiments. Recall that the representation of the total flow is given by (3), and the pair $(\phi, \tilde{h})$ is represented by (10). The eigenfunctions $\zeta$ and $\xi$ have to be found numerically. The flow region is determined implicitly by the condition $h = 1 - y^2/2 + \tilde{h} > 0$.

We take a closer look as the terms governing the evolution of $g$ and their physical significance. We note that the adjoint eigenfunction $f$ has the form

$$(f, (\phi, h)) = \frac{1}{2\sqrt{2}L} \int_0^L \int_{-\sqrt{2}}^{\sqrt{2}} h\, dy\, dx, \tag{18}$$

and with $u = (\phi, h)$, $v = (\tilde{\phi}, \tilde{h})$, we find further that

$$(f, N(u, v)) = -\frac{1}{4L\sqrt{2}} \int_0^L \left(h \frac{\partial \tilde{\phi}}{\partial y} + \tilde{h} \frac{\partial \phi}{\partial y}\right)(x, \sqrt{2}) - \left(h \frac{\partial \tilde{\phi}}{\partial y} + \tilde{h} \frac{\partial \phi}{\partial y}\right)(x, -\sqrt{2})\, dx. \tag{19}$$

Let now $\zeta = \exp(i\alpha_0 x)(\phi_1(y), h_1(y))$, $\xi = \exp(i\alpha_0 x)(\phi_2(y), h_2(y))$. Then the equation (5), evaluated at $y = \pm\sqrt{2}$ yields that

$$i\alpha_0 y h_1 - y \frac{\partial \phi_1}{\partial y} = 0, \quad h_1 + i\alpha_0 y h_2 - y \frac{\partial \phi_2}{\partial y} = 0. \tag{20}$$

By using these relationships, we can verify that

$$(f, N(\zeta, \bar{\zeta})) = 0, \quad (f, N(\zeta, \bar{\xi})) = -\frac{1}{8}(|h_1|^2(\sqrt{2}) + |h_1|^2(-\sqrt{2})). \tag{21}$$

We now return to the original, fully nonlinear problem. Let $m(x, t)$ and $M(x, t)$ denote the bounds of the flow domain, i.e. we have $h(x, m(x, t), t) = h(x, M(x, t), t) = 0$. It is clear that the total volume of the fluid,

$$\int_0^L \int_{m(x,t)}^{M(x,t)} h(x, y, t)\, dy\, dx \tag{22}$$

is a constant of the motion. If we now let $h = H + \tilde{h}$, with $\tilde{h}$, small, we can linearize the equation $h(x, m(x, t), t) = 0$ in the form

$$H'(-\sqrt{2})(m(x, t) + \sqrt{2}) + h(x, -\sqrt{2}, t) = 0, \tag{23}$$

and analogously for $M(x,t)$. Moreover, on the interval between $-\sqrt{2}$ and $m$, we can approximate $h$ as $H'(-\sqrt{2})(y-\sqrt{2})+h(x,-\sqrt{2},t)$. The resulting approximation of (22) takes the form

$$\int_0^L \int_{-\sqrt{2}}^{\sqrt{2}} h\,dy\,dx - \frac{1}{2}\int_0^L \frac{|h(x,\sqrt{2},t)|^2}{H'(\sqrt{2})} - \frac{|h(x,-\sqrt{2},t)|^2}{H'(-\sqrt{2})}\,dx. \tag{24}$$

The equation for $g$ above is just this relationship, appropriately truncated. Moreover, the particular choice of $g$ made above simply means that the perturbed flow has the same total fluid volume as the unperturbed base flow.

### 3. Reversible Hopf bifurcation with $O(2)$ symmetry

We now consider the case where the critical eigenmode does not have reflection symmetry, as e.g. for the resonance which occurs near $\alpha = 1.916$. At $L = L_0$, the operator $A(L_0)$ now has eigenvalues $\pm i\omega_0$, and the eigenvalue $i\omega_0$ has two corresponding eigenfunctions $\zeta_1 = u_1(y)\exp(i\hat{x})$ and $\zeta_2 = u_1(-y)\exp(-i\hat{x})$. Again, the eigenvalues are not semisimple, and we have generalized eigenfunctions $\xi_i$ such that $(A(L_0) - i\omega_0)\xi_i = \zeta_i$. In analogous notation as before, we denote the adjoint eigenfunctions by $c_i$, the generalized adjoint eigenfunctions by $d_i$, and we again normalize such that $(c_i, \xi_i) = 1$. The ansatz analogous to (10) is now

$$\begin{aligned} u = {} & \varepsilon\left[e^{i\omega_0 t}(a_1(\varepsilon t)\zeta_1 + a_2(\varepsilon t)\zeta_2) + e^{-i\omega_0 t}(\overline{a}_1(\varepsilon t)\overline{\zeta}_1 + \overline{a}_2(\varepsilon t)\overline{\zeta}_2)\right] \\ & + \varepsilon^2\left[e^{i\omega_0 t}(b_1(\varepsilon t)\xi_1 + b_2(\varepsilon t)\xi_2) + e^{-i\omega_0 t}(\overline{b}_1(\varepsilon t)\overline{\xi}_1 + \overline{b}_2(\varepsilon t)\overline{\xi}_2)\right. \\ & + g(\varepsilon t)\eta + e^{2i\omega_0 t}(a_1^2\psi_1 + a_2^2\psi_2 + a_1a_2\psi_3) + e^{-2i\omega_0 t}(\overline{a}_1^2\overline{\psi}_1 + \overline{a}_2^2\overline{\psi}_2 + \overline{a}_1\overline{a}_2\overline{\psi}_3) \\ & \left. + a_1\overline{a}_1\chi_1 + a_2\overline{a}_2\chi_2 + a_1\overline{a}_2\chi_3 + a_2\overline{a}_1\overline{\chi}_3\right] + O(\varepsilon^3). \end{aligned} \tag{25}$$

We obtain the following equations, analogous to (13) above,

$$\begin{aligned} \ddot{a}_1 &= \gamma a_1 + \kappa_1|a_1|^2 + \kappa_2|a_2|^2 a_1 + \nu a_1 g, \\ \ddot{a}_2 &= \gamma a_2 + \kappa_1|a_2|^2 a_2 + \kappa_2|a_1|^2 a_2 + \nu a_2 g, \\ \dot{g} &= \mu(a_1\dot{\overline{a}}_1 + \dot{a}_1\overline{a}_1 + a_2\dot{\overline{a}}_2 + \dot{a}_2\overline{a}_2). \end{aligned} \tag{26}$$

Here we have

$$\begin{aligned} \kappa_1 &= 2(c_1, N(\psi_1,\overline{\zeta}_1)) + 2(c_1, N(\chi_1,\zeta_1)) = 2(c_2, N(\psi_2,\overline{\zeta}_2)) + 2(c_2, N(\chi_2,\zeta_2)), \\ \kappa_2 &= (c_1, N(\psi_3,\overline{\zeta}_2)) + 2(c_1, N(\chi_2,\zeta_1)) + 2(c_1, N(\chi_3,\zeta_2)) \\ &= 2(c_2, N(\psi_3,\overline{\zeta}_1)) + 2(c_2, N(\chi_1,\zeta_2)) + 2(c_2, N(\overline{\chi}_3,\zeta_1)), \\ \gamma &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}(c_1, (A(L) - A(L_0))\zeta_1) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}(c_2, (A(L) - A(L_0))\zeta_2), \\ \nu &= 2(c_1, N(\eta,\zeta_1)) = 2(c_2, N(\eta,\zeta_2)), \\ \mu &= 2(f, N(\zeta_1,\overline{\xi}_1)) = 2(f, N(\zeta_2,\overline{\xi}_2)). \end{aligned} \tag{27}$$

The nonlinear contributions satisfy the equations

$$\begin{aligned} (A(L_0) - 2i\omega_0)\psi_1 + N(\zeta_1,\zeta_1) &= 0, \\ (A(L_0) - 2i\omega_0)\psi_2 + N(\zeta_2,\zeta_2) &= 0, \\ (A(L_0) - 2i\omega_0)\psi_3 + N(\zeta_1,\zeta_2) &= 0, \\ A(L_0)\chi_1 + 2N(\zeta_1,\overline{\zeta}_1) = 0,\ (f,\chi_1) &= 0, \\ A(L_0)\chi_2 + 2N(\zeta_2,\overline{\zeta}_2) = 0,\ (f,\chi_2) &= 0, \\ A(L_0)\chi_3 + 2N(\zeta_1,\overline{\zeta}_2) &= 0. \end{aligned} \tag{28}$$

The symmetries of the system are spatial translation

$$(a_1, a_2, b_1, b_2, g) \to (a_1 \exp(i\phi), a_2 \exp(i\phi), b_1 \exp(i\phi), b_2 \exp(i\phi), g),$$

reflection

$$(a_1, a_2, b_1, b_2, g) \to (a_2, a_1, b_2, b_1, g),$$

and time reversibility

$$(a_1, a_2, b_1, b_2, g) \to (\bar{a}_1, \bar{a}_2, -\bar{b}_1, -\bar{b}_2, g), \ t \to -t.$$

Similarly as in the previous section, we choose $g = \mu(|a_1|^2 + |a_2|^2)$, and we find

$$\begin{aligned}\ddot{a}_1 &= \gamma a_1 + (\kappa_1 + \mu\nu)|a_1|^2 a_1 + (\kappa_2 + \mu\nu)|a_2|^2 a_1,\\ \ddot{a}_2 &= \gamma a_2 + (\kappa_1 + \mu\nu)|a_2|^2 a_2 + (\kappa_2 + \mu\nu)|a_1|^2 a_1.\end{aligned} \tag{29}$$

Reversibility of the system implies that the coefficients are real (time reversal is equivalent to complex conjugation of the amplitudes). We consider two types of solutions of (29): traveling waves where $a_2 = 0$ (or $a_1 = 0$) and standing waves where $a_1 = a_2$. For traveling waves, we set $a_1 = A\exp(i\omega t)$, and we find the equation

$$-\omega^2 = \gamma + (\kappa_1 + \mu\nu)|A|^2; \tag{30}$$

w.l.o.g. we can assume that $A$ is real. The linearization at the solution given by (30) yields eigenvalues given by

$$\begin{aligned}\lambda^2 &= \gamma + 3(\kappa_1 + \mu\nu)|A|^2 = -2\gamma - 3\omega^2,\\ \lambda^2 &= \gamma + (\kappa_1 + \mu\nu)|A|^2 = -\omega^2,\\ \lambda^2 &= \gamma + (\kappa_2 + \mu\nu)|A|^2 = -\omega^2 + (\kappa_2 - \kappa_1)|A|^2.\end{aligned} \tag{31}$$

For standing waves, we set $a_1 = a_2 = A\exp(i\omega t)$, which leads to

$$-\omega^2 = \gamma + (\kappa_1 + \kappa_2 + 2\mu\nu)|A|^2. \tag{32}$$

The eigenvalues of the linearization are given by

$$\begin{aligned}\lambda^2 &= \gamma + 3(\kappa_1 + \kappa_2 + 2\mu\nu)|A|^2 = -2\gamma - 3\omega^2,\\ \lambda^2 &= \gamma + (\kappa_1 + \kappa_2 + 2\mu\nu)|A|^2 = -\omega^2,\\ \lambda^2 &= \gamma + (3\kappa_1 - \kappa_2 + 2\mu\nu)|A|^2 = -\omega^2 + (\kappa_1 - \kappa_2)|A|^2,\\ \lambda^2 &= \gamma + (\kappa_1 + \kappa_2 + 2\mu\nu)|A|^2 = -\omega^2.\end{aligned} \tag{33}$$

Numerically, we find the values $\kappa_1 + \mu\nu = -1.15$, $\kappa_2 + \mu\nu = 11.1$. We can conclude from this that traveling waves bifurcate supercritically, while standing waves bifurcate subcritically. The supercritical traveling waves are unstable, due to the last pair of eigenvalues in (31).

We now give a partial discussion of more general solutions of (29). The system (29) is Hamiltonian, with the Hamiltonian given by

$$\frac{1}{2}(|\dot{a}_1|^2 + |\dot{a}_2|^2) - \frac{\gamma}{2}(|a_1|^2 + |a_2|^2) - \frac{\kappa_1 + \mu\nu}{4}(|a_1|^4 + |a_2|^4) - \frac{\kappa_2 + \mu\nu}{2}|a_1|^2 + |a_2|^2. \tag{34}$$

In addition, there are the integrals of motion $i(|\bar{a}_1\dot{a}_1 - a_1\dot{\bar{a}}_1)$ and $i(|\bar{a}_2\dot{a}_2 - a_2\dot{\bar{a}}_2)$. Although we cannot integrate the system completely, we can solve it partially. Let $a_1 = r_1\exp(i\phi_1)$ and $a_2 = r_2\exp(i\phi_2)$; then $r_i^2\dot{\phi}_i$ is an integral of the motion: $r_i^2\dot{\phi}_i = C_i$. For the $r_i$, we obtain the reduced system

$$\begin{aligned} \ddot{r}_1 &= \frac{c_1^2}{r_1^3} + \gamma r_1 + (\kappa_1 + \mu\nu) r_1^3 + (\kappa_2 + \mu\nu) r_2^2 r_1, \\ \ddot{r}_2 &= \frac{c_2^2}{r_2^3} + \gamma r_2 + (\kappa_1 + \mu\nu) r_2^3 + (\kappa_2 + \mu\nu) r_1^2 r_2. \end{aligned} \tag{35}$$

It does not appear possible to find the general solution of this system, but we can identify "traveling wave" solutions for which either $r_1$ or $r_2$ vanishes, as well as "standing wave" solutions for which $r_1 = r_2$ (and $C_1 = C_2$). For either class of solutions, the reduced system is analogous to the reversible Hopf bifurcation, and hence the results of [4] carry over. Another class of solutions which we can find explicitly are those for which $\dot{r}_1 = \dot{r}_2 = 0$; these are quasiperiodic solutions which combine opposite traveling waves of unequal amplitudes.

A normal form for higher order corrections to the system can be derived along the lines of [1] and [4], provided that no resonances occur. If the analysis there is adapted to the current situation, the resulting normal form is

$$\begin{aligned} \dot{a}_1 &= b_1 + ia_1 f(L, |a_1|^2, i(a_1\bar{b}_1 - b_1\bar{a}_1), |a_2|^2, i(a_2\bar{b}_2 - b_2\bar{a}_2)), \\ \dot{b}_1 &= b_1 f(L, |a_1|^2, i(a_1\bar{b}_1 - b_1\bar{a}_1), |a_2|^2, i(a_2\bar{b}_2 - b_2\bar{a}_2)) \\ &\quad + ia_1 g(L, |a_1|^2, i(a_1\bar{b}_1 - b_1\bar{a}_1), |a_2|^2, i(a_2\bar{b}_2 - b_2\bar{a}_2)), \\ \dot{a}_2 &= b_2 + ia_2 f(L, |a_2|^2, i(a_2\bar{b}_2 - b_2\bar{a}_2), |a_1|^2, i(a_1\bar{b}_1 - b_1\bar{a}_1)), \\ \dot{b}_2 &= b_2 f(L, |a_2|^2, i(a_2\bar{b}_2 - b_2\bar{a}_2), |a_1|^2, i(a_1\bar{b}_1 - b_1\bar{a}_1)) \\ &\quad + ia_2 g(L, |a_2|^2, i(a_2\bar{b}_2 - b_2\bar{a}_2), |a_1|^2, i(a_1\bar{b}_1 - b_1\bar{a}_1)). \end{aligned} \tag{36}$$

If we consider "traveling waves" ($a_2 = b_2 = 0$) or "standing waves" ($a_1 = a_2, b_1 = b_2$), then this system is the same as system (3) in [4] (the quantities called $A$ and $B$ there are $\exp(i\omega_0 t)$ times our $a$ and $b$). As was pointed out earlier, the analysis in this paper is purely formal, and we cannot make any claim about persistence of solutions for the full system.

## 4. Conclusions

We have analyzed the local bifurcations resulting from the instability of parallel currents in a rotating system; the current is bounded by a density front on both sides. Earlier work has shown this system to be linearly unstable to disturbances of certain wavelenghts. If periodic boundary conditions are assumed and the period is treated as a bifurcation parameter, then a reversible Hopf bifurcation results. We have derived the amplitude equations at leading order, and we have evaluated the coefficients in these amplitude equations by numerical methods. The signs of these coefficients are such that there are no stable bifurcating solutions. It is hence to be expected that the instability will grow to large amplitude, perhaps to the point where meanders lead to an eventual breakup of the current. This conclusion is in agreement with the experiments of [2].

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